Let $\gamma : [0,1] \longrightarrow Y$ be a path with

$\gamma(0) = \underbrace{(0,0)}_{\overset{\circ}{=} P}$. Then we claim γ is constant, i.e.,

$$\gamma^{-1}\big(\underset{P}{(0,0)}\big) = A = \{ t \in [0,1] ;\ \gamma(t) = \underset{P}{(0,0)} \} = [0,1]$$

In other words, no point in Y is connected to $\underset{P}{(0,0)}$ via a path (other than $\underset{P}{(0,0)}$)

We can write $\gamma(t) = (\alpha(t), \tilde{f}(\alpha(t)))$

where $\tilde{f}(x) = \begin{cases} \cos(1/x) & x > 0 \\ 0 & x = 0 \end{cases}$

and

Projection onto "x-axis"

$$\alpha : [0,1] \xrightarrow{\gamma} \underset{\mathbb{R}^2}{\overset{}{Y \atop \cap}} \xrightarrow{\pi_x} [0,+\infty)$$ (so, α is a path)

$\therefore\ A = \{ t \in [0,1] ;\ \alpha(t) = 0 \} \subset [0,1]$

$[0,1]$ is connected so, it suffices to show $\underline{A \neq \emptyset}$ and A is both open and $\underline{\text{closed}}$.

okay

$0 \in A$ and A is closed as it is the inverse image of a closed set under a continuous map

Pick $a \in A$.

Since γ is continuous at a, $\exists\, \delta > 0$ such that

$$t \in (a-\delta, a+\delta) \Rightarrow \|\gamma(t)\| < 1 = \varepsilon$$

$$\Updownarrow \qquad\qquad \|\gamma(t)\| = |\gamma(t) - \underbrace{\gamma(a)}_{=(0,0)}|$$

$$|t-a| < \delta$$

Let $J = (a-\delta, a+\delta)$. Since α is continuous, $\alpha(J)$ is also an interval (intermediate value theorem) and $0 \in \alpha(J)$. We want to show $J \subset A$ (A is open) i.e. $\Updownarrow$ $\alpha(J) = \{0\}$

Now, if $\alpha(J)$ is not degenerate then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{2\pi n} \in \alpha(J)$

$$\Updownarrow$$

$$\exists\, t \in J \text{ s.t. } \alpha(t) = \frac{1}{2\pi n}$$

$$\Updownarrow$$

$$\exists\, t \in J \text{ s.t. } \gamma(t) = \left(\frac{1}{2\pi n}, \cos(2\pi n)\right)$$

but for this t $\|\gamma(t)\| > 1$?